Bertrand's Postulate

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Abstract

We prove there always exists a prime integer between n and $2n$ for any natural number $n > 1$.

1 Notation

Throughout this paper, we will use p to represent a positive prime integer. Notation such as

$$\prod_{p \leq n} p$$

is interpreted as the product of all primes less than or equal to n.

For example, the prime counting function $\pi(x)$ is the number of primes less than or equal to x. Formally, we can express this as

$$\pi(x) = \sum_{p \leq x} 1.$$

2 Lemmas

We first establish some very helpful lemmas.

Lemma 2.1. *For any natural number n, we have*

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Proof. This is a well-known but non-trivial fact. Some popular proofs of this involve the binomial theorem or induction. A nice combinatorial proof is as follows. Let S be a set of size n. We count the size of $\mathcal{P}(S)$, the power set of S (set of all subsets of S), in two ways:

1. Recall the number of subsets of size k is $\binom{n}{k}$. Thus, the number of subsets of all sizes of S can be expressed as $\sum_{k=0}^{n} \binom{n}{k}$.

2. Consider constructing all subsets of S. For each element $x \in S$, either x is included in the subset or not included in the subset. In this way, we see that the number of subsets of S is 2^n.

So, we have that

$$|\mathcal{P}(S)| = \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

□

Lemma 2.2. *(Legendre's Formula) For any natural number n, the prime factorization of $n!$ can be expressed as*

$$n! = \prod_{p \leq n} p^{e_p}, \quad \textit{where } e_p = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor.$$

Proof. We want to find the largest power of p that divides $n! = 1 \times 2 \times \cdots \times n$. To count how many times p appears in this product, we count how many numbers 1 to n are divisible by p, then by p^2, then by p^3, etc.

From 1 to n, there are $\lfloor \frac{n}{p^k} \rfloor$ numbers divisible by p^k. Legendre's Formula then easily follows. □

Lemma 2.3. *Define a function $\psi : \mathbb{R}_{\geq 0} \to \mathbb{Z}$ as $\psi(x) = \lfloor 2x \rfloor - 2\lfloor x \rfloor$ for all $x \in \mathbb{R}_{\geq 0}$. Let $\{x\} = x - \lfloor x \rfloor$ denote the "decimal part" of a real number x. Clearly, $0 \leq \{x\} < 1$. We then have*

$$\psi(x) = \begin{cases} 0 & \textit{if } 0 \leq \{x\} < \frac{1}{2}, \\ 1 & \textit{if } \frac{1}{2} \leq \{x\} < 1. \end{cases}$$

Proof. Suppose $0 \leq \{x\} < \frac{1}{2}$. Then, we have

$$\begin{aligned} 0 \leq x - \lfloor x \rfloor &< \frac{1}{2}, \\ \lfloor x \rfloor \leq x &< \lfloor x \rfloor + \frac{1}{2}, \\ 2\lfloor x \rfloor \leq 2x &< 2\lfloor x \rfloor + 1. \end{aligned}$$

From the first inequality, note that $2\lfloor x \rfloor$ is an integer, and so we must also have $2\lfloor x \rfloor \leq \lfloor 2x \rfloor$, implying $\psi(x) \geq 0$. From the second inequality, we can write $\lfloor 2x \rfloor < 2\lfloor x \rfloor + 1$ and see that $\psi(x) < 1$. Since we know $\psi(x)$ must be an integer, we must have $\psi(x) = 0$.

Suppose $\frac{1}{2} \leq \{x\} < 1$. Then, we have

$$\begin{aligned} \frac{1}{2} &\leq x - \lfloor x \rfloor < 1, \\ \lfloor x \rfloor + \frac{1}{2} &\leq x < \lfloor x \rfloor + 1, \\ 2\lfloor x \rfloor + 1 &\leq 2x < 2\lfloor x \rfloor + 2. \end{aligned}$$

From the first inequality, since $2\lfloor x \rfloor + 1$ is an integer, we must also have $2\lfloor x \rfloor + 1 \leq \lfloor 2x \rfloor$, implying $\psi(x) \geq 1$. From the second inequality, we can write $\lfloor 2x \rfloor < 2\lfloor x \rfloor + 2$ and see that $\psi(x) < 2$. Thus, we must have $\psi(x) = 1$. □

Lemma 2.4. *For all real numbers $x \geq 1$, we have $\pi(x) \leq x - 1$.*

Proof. Clearly, the number of positive primes less than or equal to x is upper-bounded by the number of positive integers less than or equal to x. There are $\lfloor x \rfloor$ integers less than or equal to x, and 1 is always among those integers. Since 1 is not prime, the number of primes is further upper-bounded by $\lfloor x \rfloor - 1$. The lemma easily follows. □

Lemma 2.5. *For all real numbers $x \geq 1$, we have that*

$$\prod_{p \leq x} p \leq 4^x.$$

Proof. It is sufficient to show this lemma is true for all $x \in \mathbb{N}$, since between integers, the left-hand side doesn't change while the right-hand side clearly increases. We prove by induction.

Our base cases of $x = 1, 2$ hold by simple computation. Assume the lemma holds for $x = 1, 2, \ldots, n-1$. We show this implies the lemma also holds for $x = n$.

If n is even, then n cannot be prime. So,

$$\prod_{p \leq n} p = \prod_{p \leq n-1} p \leq 4^{n-1} \leq 4^n,$$

and the lemma holds true. If n is odd, write $n = 2k + 1$ for some integer k. Now, note that $\binom{2k+1}{k} = \frac{(2k+1)!}{k!(k+1)!}$ is divisible by every prime p with $(k+2) \leq p \leq 2k+1$. So,

$$\prod_{k+2 \leq p \leq 2k+1} p \mid \binom{2k+1}{k}.$$

We know that $\binom{2k+1}{k} = \binom{2k+1}{k+1}$, and from Lemma 2.1, we know that $\binom{2k+1}{k} + \binom{2k+1}{k+1} \leq 2^{2k+1}$ is certainly true. Combining these gives $2\binom{2k+1}{k} \leq 2^{2k+1}$, implying $\binom{2k+1}{k} \leq 2^{2k}$. So, we have

$$\prod_{k+2 \leq p \leq 2k+1} p \leq \binom{2k+1}{k} \leq 2^{2k} = 4^k.$$

Thus, we can write

$$\prod_{p \leq 2k+1} = \left(\prod_{p \leq k+1} p \right) \left(\prod_{k+2 \leq p \leq 2k+1} p \right) \leq (4^{k+1})(4^k) = 4^{2k+1}.$$

We have shown the lemma holds true for $x = n$, so by induction, the lemma holds true for all $x \in \mathbb{N}$. □

Lemma 2.6. *Bertrand's Postulate holds for $n \leq 630$. Explicitly, for every natural number $1 < n \leq 630$, there exists a prime number p such that $n < p < 2n$.*

Proof. The proof follows from the list of manually verified prime numbers below:

$$2, 3, 5, 7, 13, 23, 43, 83, 163, 317, 631.$$

□

3 Bertrand's Postulate

Theorem 3.1. *(Bertrand's Postulate) For any natural number $n > 1$, there exists a prime p such that $n < p < 2n$.*

Proof. Let n be a natural number with $n > 1$. We analyze the middle binomial coefficient $\binom{2n}{n}$. We will derive lower and upper bounds on $\binom{2n}{n}$ and show that Bertrand's Postulate must be true for the bounds to hold.

For the lower bound, we observe that $\binom{2n}{k}$ is maximized when $k = n$. Thus, $\binom{2n}{n} \geq \binom{2n}{k}$ for $0 \leq k \leq 2n$.

Define the set S as

$$S = \left\{ \binom{2n}{0} + \binom{2n}{2n}, \binom{2n}{1}, \binom{2n}{2}, \dots, \binom{2n}{2n-1} \right\}.$$

Note that since $n > 1$, we must have $\binom{2n}{n} > 2$. Since $\binom{2n}{0} = \binom{2n}{2n} = 1$, we see that $\binom{2n}{n} \geq s$ for all $s \in S$. Thus, it must be the case that $\binom{2n}{n}$ is lower bound of the average of all elements in S. With Lemma 2.1, we have that

$$\binom{2n}{n} \geq \frac{1}{2n} \sum_{s \in S} s = \frac{1}{2n} \sum_{k=0}^{2n} \binom{2n}{k} = \frac{2^{2n}}{2n} \tag{1}$$

For the upper bound, note that $\binom{2n}{n} = \frac{(2n)!}{n!n!}$, which we can use Lemma 2.2 to prime factorize as

$$\binom{2n}{n} = \frac{\prod_{p \leq 2n} p^{e_p}}{(\prod_{p \leq n} p^{f_p})^2} = \prod_{p \leq 2n} p^{e_p - 2f_p}, \quad \text{where } e_p = \sum_{k=1}^{\infty} \left\lfloor \frac{2n}{p^k} \right\rfloor \text{ and } f_p = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor.$$

Recall the ψ function defined in Lemma 2.3. We can rewrite the exponent $e_p - 2f_p$ as

$$e_p - 2f_p = \sum_{k=1}^{\infty} \left\lfloor \frac{2n}{p^k} \right\rfloor - 2 \left\lfloor \frac{n}{p^k} \right\rfloor = \sum_{k=1}^{\infty} \psi\left(\frac{n}{p^k}\right).$$

On the last summation, note that for $k > \log_p(2n)$, we have that $p^k > 2n$ implying $0 < \frac{n}{p^k} < \frac{1}{2}$ and thus $\psi\left(\frac{n}{p^k}\right) = 0$. Thus, since the maximum output of the ψ function is 1, we can more specifically write

$$e_p - 2f_p = \sum_{k=1}^{\lfloor \log_p(2n) \rfloor} \psi\left(\frac{n}{p^k}\right) \leq \log_p(2n). \tag{2}$$

Let us break up the prime factorization of $\binom{2n}{n}$ into four different sections in the following way:

$$\begin{aligned}\binom{2n}{n} &= \left(\prod_{p \leq 2n} p^{e_p - 2f_p}\right) \\ &= \left(\prod_{p \leq \sqrt{2n}} p^{e_p - 2f_p}\right)\left(\prod_{\sqrt{2n} < p \leq \frac{2n}{3}} p^{e_p - 2f_p}\right)\left(\prod_{\frac{2n}{3} < p \leq n} p^{e_p - 2f_p}\right)\left(\prod_{n < p \leq 2n} p^{e_p - 2f_p}\right).\end{aligned}$$

For primes $p > \sqrt{2n}$, we have that $p^2 > 2n$, implying $\frac{n}{p^2} < \frac{1}{2}$. From this, we see $\frac{n}{p^k} < \frac{1}{2}$ for all $k \geq 2$. With Lemma 2.3, we see that $\psi(\frac{n}{p^k}) = 0$ for $k \geq 2$, and

$$e_p - 2f_p = \psi\left(\frac{n}{p}\right), \quad \text{for } p > \sqrt{2n}.$$

For primes $n < p < 2n$, we have that $\frac{1}{2} < \frac{n}{p} < 1$. It then follows that $\left\{\frac{n}{p}\right\} \geq \frac{1}{2}$ and so $\psi\left(\frac{n}{p}\right) = 1$. (As a sanity check, note that primes between n and $2n$ must appear in the prime factorization of $(2n)!$ but cannot appear in the prime factorization of $n!$. Thus, it makes sense these primes remain in the prime factorization of $\binom{2n}{n} = \frac{(2n)!}{n!n!}$.)

For primes $\frac{2n}{3} < p \leq n$, we have that $1 \leq \frac{n}{p} < \frac{3}{2}$, implying that $\{\frac{n}{p}\} < \frac{1}{2}$. So, $\psi\left(\frac{n}{p}\right) = 0$.

Now, we can write

$$\begin{aligned}\binom{2n}{n} &= \left(\prod_{p \leq \sqrt{2n}} p^{e_p - 2f_p}\right)\left(\prod_{\sqrt{2n} < p \leq \frac{2n}{3}} p^{\psi(\frac{n}{p})}\right)\left(\prod_{\frac{2n}{3} < p \leq n} p^0\right)\left(\prod_{n < p \leq 2n} p\right) \\ &= \left(\prod_{p \leq \sqrt{2n}} p^{e_p - 2f_p}\right)\left(\prod_{\sqrt{2n} < p \leq \frac{2n}{3}} p^{\psi(\frac{n}{p})}\right)\left(\prod_{n < p \leq 2n} p\right).\end{aligned}$$

Recalling the maximum output of the ψ function is 1 and (2), we have

$$\begin{aligned}
\binom{2n}{n} &= \left(\prod_{p\le\sqrt{2n}} p^{e_p-2f_p}\right)\left(\prod_{\sqrt{2n}<p\le\frac{2n}{3}} p^{\psi(\frac{n}{p})}\right)\left(\prod_{n<p\le 2n} p\right) \\
&\le \left(\prod_{p\le\sqrt{2n}} p^{\log_p(2n)}\right)\left(\prod_{\sqrt{2n}<p\le\frac{2n}{3}} p\right)\left(\prod_{n<p\le 2n} p\right) \\
&= \left(\prod_{p\le\sqrt{2n}} 2n\right)\left(\prod_{\sqrt{2n}<p\le\frac{2n}{3}} p\right)\left(\prod_{n<p\le 2n} p\right) \\
&= (2n)^{\pi(\sqrt{2n})}\left(\prod_{\sqrt{2n}<p\le\frac{2n}{3}} p\right)\left(\prod_{n<p\le 2n} p\right).
\end{aligned}$$

Applying Lemma 2.4 and being a bit less restrictive on some bounds gives a final upper bound of

$$\binom{2n}{n} \le (2n)^{\sqrt{2n}-1}\left(\prod_{p\le\frac{2n}{3}} p\right)\left(\prod_{n<p\le 2n} p\right). \tag{3}$$

Now, suppose Bertrand's Postulate is false, and there is some n such that no primes exist between n and $2n$. The last product in the upper bound equation above would collapse to 1. Combining the lower bound in (1), we have

$$\frac{2^{2n}}{2n} \le \binom{2n}{n} \le (2n)^{\sqrt{2n}-1}\left(\prod_{p\le\frac{2n}{3}} p\right).$$

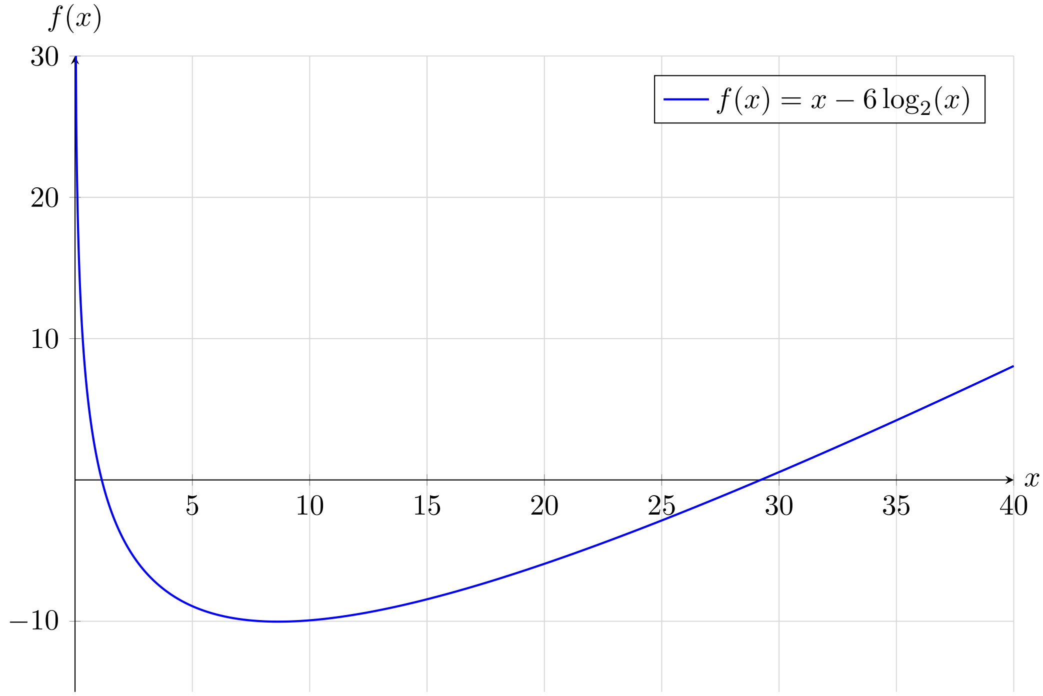
Applying Lemma 2.5 to the product, we have

$$\begin{aligned}
\frac{2^{2n}}{2n} &\le (2n)^{\sqrt{2n}-1}4^{\frac{2n}{3}} = (2n)^{\sqrt{2n}-1}2^{\frac{4n}{3}}, \\
2^{\frac{2n}{3}} &\le (2n)^{\sqrt{2n}}, \\
2^{\frac{\sqrt{2n}}{3}} &\le 2n.
\end{aligned}$$

Let $x = \sqrt{2n}$. We have that

$$\begin{aligned}
2^{\frac{x}{3}} &\le x^2, \\
\frac{x}{3} &\le 2\log_2(x), \\
x &\le 6\log_2(x), \\
x - 6\log_2(x) &\le 0.
\end{aligned}$$

Let $f(x) = x - 6\log_2(x)$. Graphing technology shows that $f(x) \leq 0$ certainly does not hold true for $x \geq 32$:



We can reach the same conclusion without technology. Note that $f'(x) = 1 - \frac{6}{x\ln(2)}$. This function is increasing when $f'(x) \geq 0$, or when

$$1 - \frac{6}{x\ln(2)} \geq 0,$$
$$x \geq \frac{6}{\ln(2)} \approx 8.65...$$

Plugging in $x = 32$ gives $f(32) = 32 - 6\log_2(32) = 32 - 30 = 2$. Since x only increases from here, it's clear that $f(x) > 0$ for $x \geq 32$.

We have found a contradiction for $x \geq 32$. Note $x = \sqrt{2n} \geq 32$ implies $n \geq 512$. Thus, we have found a contradiction for $n \geq 512$, showing Bertrand's Postulate must hold true for such n. Combined with Lemma 2.6, the entirety of Bertrand's Postulate follows. $\square$